Lecture 1: From Quantum Mechanics to Relativistic Quantum Mechanics

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Introductory Lectures on the Standard Model

Purpose of Lecture 1

Central question

Why is ordinary non-relativistic quantum mechanics not the right language for elementary particles?

$$\textbf{QM} + \textbf{special relativity} \Longrightarrow \textbf{relativistic wave equations}$$

But these equations will reveal a deeper problem:

$$\boxed{\text{a fixed-particle-number wavefunction is not enough.}}$$

This prepares the transition to quantum field theory in Lecture 2.

Prolegomenon: why this route?

Questions to motivate the course:

- Why does non-relativistic quantum mechanics work so well at low energies?
- What goes wrong when relativity becomes important?
- Why do antiparticles appear?
- Why does spin appear naturally?
- Why does the photon have only two physical polarizations?
- How do we convert amplitudes into decay rates and cross sections?

Lecture 1 ends just before QFT becomes unavoidable.

Roadmap of the lecture

1. Non-relativistic quantum mechanics and probability conservation
2. Why particle physics stresses ordinary quantum mechanics
3. Klein–Gordon equation
4. Dirac equation
5. Maxwell equations and photon degrees of freedom
6. Transition rates and cross sections

$$\boxed{\text{Schrödinger} \to \text{Klein–Gordon} \to \text{Dirac} \to \text{Maxwell} \to \Gamma, \sigma}$$

The Schrödinger equation

Start from the free-particle Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m}.$$

we postulate Schrödinger's equation[1]

$$\boxed{-\frac{1}{2m}\nabla^2\psi \;=\; i\,\frac{\partial\psi}{\partial t}\;.}$$

Plane-wave solutions:

$$\psi(x) \sim e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)}, \qquad E = \frac{\boldsymbol{p}^2}{2m}.$$

This is excellent for low-energy fixed-particle-number physics.

[1] I will use units with $\hbar = c = 1$. Sorry for that!

Probability density and current

The probabilistic interpretation uses

$$\int d^3x \, |\psi(\mathbf{x}, t)|^2 < \infty \, , \qquad \rho = |\psi|^2.$$

The Schrödinger equation implies

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0,}$$

with

$$\boxed{\boldsymbol{j} = -\frac{i}{2m} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right).}$$

Integrating over space,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3x \, |\psi|^2 = 0.$$

$\boxed{\text{Total probability is conserved.}}$

The hidden assumption: fixed particle number

The wavefunction $\psi(\boldsymbol{x}, t)$ describes one particle.

For many particles, ordinary quantum mechanics uses

$$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N, t),$$

with fixed N.

But particle physics has processes such as

$$\gamma + \gamma \to e^+e^-, \qquad Z \to e^+e^-, \qquad e^+e^- \to \mu^+\mu^-, \quad \text{etc.}$$

$\boxed{\text{Relativistic particle physics requires particle creation and annihilation.}}$

This is not naturally described by a fixed-particle-number wavefunction.

First attempt: make quantum mechanics relativistic

Use the relativistic energy-momentum relation

$$E^2 = \boldsymbol{p}^2 + m^2.$$

we postulate

$$\boxed{(\Box + m^2)\phi = 0, \qquad \Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2.}$$

This is the **Klein–Gordon** equation.

Question:

Can this be a one-particle relativistic wave equation?

Klein–Gordon plane waves

For

$$\phi(x) = Ae^{-ip\cdot x},$$

the Klein–Gordon equation gives

$$p^2 = m^2.$$

Therefore

$$\boxed{E = \pm\sqrt{\boldsymbol{p}^2 + m^2}.}$$

Two immediate puzzles:

- Why do we have negative-energy solutions?
- What is the correct probability density?

Relativity has already doubled the spectrum.

Klein–Gordon current

The Klein–Gordon equation admits a conserved current:

$$\boxed{j^\mu = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right), \qquad \partial_\mu j^\mu = 0.}$$

The time component is

$$j^0 = i\left(\phi^* \dot{\phi} - \phi \dot{\phi}^*\right).$$

For

$$\phi = A e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{x}},$$

one obtains

$$\boxed{j^0 = 2E|A|^2.}$$

The probability-density problem

For positive-energy modes,

$$E > 0 \quad \Rightarrow \quad j^0 > 0.$$

But for negative-energy modes,

$$E < 0 \quad \Rightarrow \quad j^0 < 0.$$

Therefore

$$\boxed{j^0 \text{ is not positive definite.}}$$

So the Klein–Gordon current cannot be interpreted as an ordinary probability current for a one-particle wavefunction.

Important lesson

The Klein–Gordon equation is not wrong. It is not a satisfactory one-particle relativistic quantum mechanics.

What did we learn from Klein–Gordon?

The Klein–Gordon equation taught us two things:

1. Relativity naturally produces both positive- and negative-frequency solutions.
2. A naive one-particle probability interpretation fails.

The KG equation will later become meaningful as a field equation.

But in Lecture 1 we continue looking for a relativistic wave equation.

Next attempt:

Find an equation first order in time.

Dirac's idea: linearize the relation

Dirac wanted an equation first order in time and space:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0.}$$

To reproduce

$$E^2 = \boldsymbol{p}^2 + m^2,$$

the matrices γ^μ must satisfy

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.}$$

This is the Clifford algebra. The γ^μ must be 4×4 matrices.

$\boxed{\psi \text{ is not a scalar wavefunction: it is a spinor.}}$

Dirac implies Klein–Gordon

Starting from

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

act with the conjugate operator:

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

Using

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

we get

$$\boxed{(\Box + m^2)\psi = 0}$$

for every component of ψ.

Lesson

The Dirac equation is a "square root" of the Klein–Gordon equation.

Dirac current

Define "psi-bar"

$$\bar{\psi} = \psi^\dagger \gamma^0.$$

The Dirac current is

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0.}$$

Its time component is

$$\boxed{j^0 = \psi^\dagger\psi \geq 0.}$$

This looks better than Klein–Gordon!

But the energy spectrum still contains

$$\boxed{E = \pm\sqrt{\boldsymbol{p}^2 + m^2}.}$$

Spin and magnetic moment

The Dirac equation naturally describes spin-$\frac{1}{2}$ particles.

In an electromagnetic field, it predicts the magnetic interaction of the electron with approximately

$$\boxed{g = 2}$$

at leading order.

This is a major success:

$$\boxed{\text{spin and magnetic moment emerge from relativistic quantum mechanics.}}$$

But the negative-energy problem remains.

Antiparticles

The Dirac equation again gives

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}.$$

Historically, this led to the Dirac sea interpretation.

Modern lesson:

$$\boxed{\text{relativistic quantum theory predicts antiparticles.}}$$

For the electron:

$$\boxed{e^- \longleftrightarrow e^+.}$$

But a fully satisfactory interpretation requires QFT.

Status after Klein–Gordon and Dirac

We have gained much:

- relativistic dispersion relation,
- spinors,
- spin-$\frac{1}{2}$,
- antiparticles,
- conserved currents.

But the one-particle picture is under stress:

$$\boxed{\text{negative-energy solutions point beyond relativistic wave mechanics.}}$$

Next: photons and Maxwell equations.

Maxwell equations in covariant form

Introduce the four-potential (reminder from Griffith's ED-book, Ch.12)

$$A^\mu = (V, \boldsymbol{A})$$

and the field-strength tensor

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.}$$

Maxwell's equations become

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu \ , \quad \text{where} \quad J^\mu = (\rho, \mathbf{J}) \ .}$$

Current conservation follows:

$$\boxed{\partial_\nu J^\nu = 0.}$$

Gauge invariance of electromagnetism

The same electromagnetic fields are obtained from

$$\boxed{A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda.}$$

because

$$F_{\mu\nu} \to F_{\mu\nu}.$$

Thus electromagnetism contains redundancy:

$$\boxed{\text{not all components of } A_\mu \text{ are physical.}}$$

This is the first appearance of gauge invariance in the course.

Photon degrees of freedom

In Lorenz gauge,

$$\partial_\mu A^\mu = 0,$$

Maxwell's equations in vacuum become

$$\boxed{\Box A^\mu = 0.}$$

This is a massless wave equation. There is still gauge invariance remain. For example, choose Coulomb gauge: $A^0 = 0$ or ...

A four-vector has four components, but gauge invariance removes unphysical components.

The photon has

$$\boxed{\text{2 physical polarizations.}}$$

$$\boxed{\text{The photon is a massless spin-1 particle.}}$$

Where Lecture 1 has brought us

By now, relativistic quantum mechanics has revealed:

- scalar relativistic wave equations,
- spinor relativistic wave equations,
- antiparticles,
- massless vector EM waves,
- gauge redundancy,
- physical polarizations.

But particle physics is experimental: we must predict rates and cross sections.

$\boxed{\text{Next: from amplitudes to measurable quantities.}}$

From amplitudes to probabilities

In particle physics the central object is the transition amplitude

$$\boxed{\mathcal{M}_{i\to f}.}$$

Observable probabilities are built from

$$\boxed{|\mathcal{M}_{i\to f}|^2.}$$

The two most important observables are

$$\boxed{\Gamma = \text{decay rate}, \qquad \sigma = \text{scattering cross section.}}$$

Lecture 1 introduces the kinematic framework needed to relate $\mathcal{M}$ to Γ and σ.

Decay rates: general structure

For a decay

$$A \to 1 + 2 + \cdots + n,$$

the rate has the schematic form

$$\boxed{\Gamma = \frac{1}{2m_A} \int \mathrm{d}\Pi_n \, |\mathcal{M}|^2.}$$

Here $\mathrm{d}\Pi_n$ is Lorentz-invariant final-state phase space.[2]

The message:

$$\boxed{\text{dynamics is in } \mathcal{M}, \qquad \text{kinematics is in phase space } d\Pi_n.}$$

[2] See Peskin's book, Ch. 7 or Griffiths's book Ch. 6 for the definition of $d\Pi_n$.

Two-body decay formula

For

$$A \to 1 + 2,$$

in the rest frame of A,

$$\boxed{\Gamma = \frac{|\boldsymbol{p}|}{32\pi^2 m_A^2} \int \mathrm{d}\Omega \, |\mathcal{M}|^2.}$$

If $|\mathcal{M}|^2$ is independent of the angles,

$$\boxed{\Gamma = \frac{|\boldsymbol{p}|}{8\pi m_A^2} |\mathcal{M}|^2.}$$

The final-state momentum is

$$\boxed{|\boldsymbol{p}| = \frac{1}{2m_A} \sqrt{[m_A^2 - (m_1 + m_2)^2][m_A^2 - (m_1 - m_2)^2]}.}$$

Scattering and cross sections

For a process

$$a + b \to 1 + 2 + \cdots + n,$$

the cross section has the schematic form

$$\boxed{\sigma = \frac{1}{\text{flux}} \int \mathrm{d}\Pi_n \, |\mathcal{M}|^2.}$$

For two-to-two scattering $a + b \to 1 + 2$, the cross-section is

$$\boxed{\sigma(a + b \to 1 + 2) = \frac{|\mathbf{p_1}|}{64\pi^2 |\mathbf{p}_a| s} \int |\mathcal{M}|^2 \, d\Omega}$$

Mandelstam variables

For

$$a + b \to 1 + 2,$$

the Mandelstam variables are

$$\boxed{s = (p_a + p_b)^2, \qquad t = (p_a - p_1)^2, \qquad u = (p_a - p_2)^2.}$$

They satisfy

$$\boxed{s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2.}$$

In the centre-of-mass frame, s is the total energy squared:

$$\boxed{\sqrt{s} = E_{\mathrm{cm}}.}$$

At the LHC, $\sqrt{s} = 13$ TeV (Run II phase).

Summary of Lecture 1

1. Schrödinger quantum mechanics conserves probability and assumes fixed particle number.
2. The Klein–Gordon equation follows from $E^2 = \boldsymbol{p}^2 + m^2$, but its density is not positive definite.
3. The Dirac equation gives spinors, spin-$\frac{1}{2}$, a positive density, and antiparticles.
4. Maxwell theory describes a massless spin-1 particle with two physical polarizations.
5. Decay rates and cross sections are obtained from $|\mathcal{M}|^2$ and phase space.

Final message

The first confrontation between quantum mechanics and special relativity already reveals the key structures of particle physics: antiparticles, spin, gauge redundancy, photons, and scattering observables. But it also shows that a one-particle wavefunction cannot be the final framework.

Next lecture:

Quantum fields, particles, interactions, and Feynman diagrams.

Reading for this lecture

- D. Griffiths, Introduction to Elementary Particle Physics, 2008.
- M. Peskin, Concepts of Elementary Particle Physics, Oxford Master Series, 2019.
- I. Aitchison and A. Hey, Gauge Theories in Particle Physics, Vol. I, IoP, 2012.
- B. Gripaios, From QM to the SM, arXiv:2005.06355

Useful on-line resources

- Where can I locate papers on elementary particle physics?

  http://inspirehep.net

- What's the best place to find daily research on elementary particle physics?

  https://arxiv.org/

- Where can I find ALL currently known properties of elementary particles? Particle Dada Group (PDG)

  http://pdg.lbl.gov

Tutorial questions and problems

1. Derive the Klein–Gordon current and show that $j^0 = 2E|A|^2$ for a plane wave.
2. Show that the Dirac equation implies the Klein–Gordon equation for each spinor component.
3. Show that $F_{\mu\nu}$ is invariant under $A_\mu \to A_\mu + \partial_\mu \Lambda$.
4. Derive the two-body decay formula for $\Gamma(A \to 1 + 2)$.
5. Show that spin emerges from Dirac's equation (**)