ISSEPP 2026

THEOREMS UNRAVELLING THE FOUNDATIONS OF QUANTUM THEORY

Andreas Zoupas

ISSEPP 2026

THEOREMS UNRAVELLING THE FOUNDATIONS OF QUANTUM THEORY

Andreas Zoupas

TOPIC

No-go theorems.

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities?

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results.

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem,

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System ”possessing” a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason’s Theorem, and

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.
- Main Idea:

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.
- Main Idea:Is it possible that probabilistic results in quantum theory have the same status as those in classical statistical physics?

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.
- Main Idea:Is it possible that probabilistic results in quantum theory have the same status as those in classical statistical physics? Namely:

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.
- Main Idea:Is it possible that probabilistic results in quantum theory have the same status as those in classical statistical physics? Namely:
    (i) Do quantum states refer to individual systems directly?

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.
- Main Idea:Is it possible that probabilistic results in quantum theory have the same status as those in classical statistical physics? Namely:
  - (i) Do quantum states refer to individual systems directly?
  - (ii) Do (perfect) measurements reveal the value that physical quantities possess immediately beforehand?

I follow closely (Isham 1995) and (Peres 2010)

TOPIC

No-go theorems.

- The Three results to follow, have, in one way or another, their roots in the so called EPR paradox
- Question: Is it meaningful, in Quantum Theory to talk about an individual System "possessing" a value for all its physical quantities? The answer to this question is mostly governed by three Results. The Kochen-Specker Theorem, Gleason's Theorem, and Bell's Inequalies.
- Main Idea:Is it possible that probabilistic results in quantum theory have the same status as those in classical statistical physics? Namely:
  - (i) Do quantum states refer to individual systems directly?
  - (ii) Do (perfect) measurements reveal the value that physical quantities possess immediately beforehand?

I follow closely (Isham 1995) and (Peres 2010)

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD):

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times:

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC):

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC): Any property (i.e. value of an observable) a QM system may possess, it does so independently of any measurement context, i.e.

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC): Any property (i.e. value of an observable) a QM system may possess, it does so independently of any measurement context, i.e. independently of how that value is eventually measured:

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC): Any property (i.e. value of an observable) a QM system may possess, it does so independently of any measurement context, i.e. independently of how that value is eventually measured: Non Contextuality.

Consequence of (VD) and (NC)

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC): Any property (i.e. value of an observable) a QM system may possess, it does so independently of any measurement context, i.e. independently of how that value is eventually measured: Non Contextuality.

Consequence of (VD) and (NC)

- There exist "hidden variables" and the statistical prediction of quantum mechanics are simply due to our ignorance of them and thus,

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC): Any property (i.e. value of an observable) a QM system may possess, it does so independently of any measurement context, i.e. independently of how that value is eventually measured: Non Contextuality.

Consequence of (VD) and (NC)

- There exist "hidden variables" and the statistical prediction of quantum mechanics are simply due to our ignorance of them and thus, QM is essentially a Classical Statistical Physics Theory.

The Kochen-Specker Theorem

Aim to Explore the Following Two Requirements:

- (VD): All observables defined for a quantum mechanical system have definite values at all times: Value Definiteness.
- (NC): Any property (i.e. value of an observable) a QM system may possess, it does so independently of any measurement context, i.e. independently of how that value is eventually measured: Non Contextuality.

Consequence of (VD) and (NC)

- There exist "hidden variables" and the statistical prediction of quantum mechanics are simply due to our ignorance of them and thus, QM is essentially a Classical Statistical Physics Theory.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-1

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-1

- Given a (topologically) closed subspace, W, and it's orthogonal complement $W^{\perp}$, of a Hilbert space $\mathscr{H}$, any vector $\vec{\psi} \in \mathscr{H}$ can be decomposed as a unique sum

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-1

- Given a (topologically) closed subspace, W, and it's orthogonal complement $W^{\perp}$, of a Hilbert space $\mathscr{H}$, any vector $\vec{\psi} \in \mathscr{H}$ can be decomposed as a unique sum

$$\vec{\psi} = \vec{\psi}_W + \vec{\psi}_{W^{\perp}}, \ \vec{\psi}_W \in W \text{ and } \vec{\psi}_{W^{\perp}} \in W^{\perp}$$

- We can then define the Operator $\hat{\mathrm{P}}_W$:

$$\hat{\mathrm{P}}_W \vec{\psi} := \vec{\psi}_W,$$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-1

- Given a (topologically) closed subspace, W, and it's orthogonal complement $W^{\perp}$, of a Hilbert space $\mathscr{H}$, any vector $\vec{\psi} \in \mathscr{H}$ can be decomposed as a unique sum

$$\vec{\psi} = \vec{\psi}_W + \vec{\psi}_{W^{\perp}},\ \vec{\psi}_W \in W \text{ and } \vec{\psi}_{W^{\perp}} \in W^{\perp}$$

- We can then define the Operator $\hat{\mathrm{P}}_W$:

$$\hat{\mathrm{P}}_W \vec{\psi} := \vec{\psi}_W,$$

the Projection operator onto the subspace W.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-1

- Given a (topologically) closed subspace, W, and it's orthogonal complement $W^{\perp}$, of a Hilbert space $\mathscr{H}$, any vector $\vec{\psi} \in \mathscr{H}$ can be decomposed as a unique sum

$$\vec{\psi} = \vec{\psi}_W + \vec{\psi}_{W^{\perp}},\ \vec{\psi}_W \in W \text{ and } \vec{\psi}_{W^{\perp}} \in W^{\perp}$$

- We can then define the Operator $\hat{\mathrm{P}}_W$:

$$\hat{\mathrm{P}}_W \vec{\psi} := \vec{\psi}_W,$$

the Projection operator onto the subspace W. Then $\hat{\mathrm{P}}_{W^{\perp}} = \hat{\mathbb{1}} - \hat{\mathrm{P}}_W$.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-1

- Given a (topologically) closed subspace, W, and it's orthogonal complement $W^{\perp}$, of a Hilbert space $\mathscr{H}$, any vector $\vec{\psi} \in \mathscr{H}$ can be decomposed as a unique sum

$$\vec{\psi} = \vec{\psi}_W + \vec{\psi}_{W^{\perp}}, \; \vec{\psi}_W \in W \text{ and } \vec{\psi}_{W^{\perp}} \in W^{\perp}$$

- We can then define the Operator $\hat{\mathrm{P}}_W$:

$$\hat{\mathrm{P}}_W \vec{\psi} := \vec{\psi}_W,$$

the Projection operator onto the subspace W. Then $\hat{\mathrm{P}}_{W^{\perp}} = \hat{\mathbb{1}} - \hat{\mathrm{P}}_W$.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-2

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-2

- Projection operator properties

$$\hat{\mathrm{P}}_W^2 = \hat{\mathrm{P}}_W$$
$$\hat{\mathrm{P}}_W^\dagger = \hat{\mathrm{P}}_W$$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-2

- Projection operator properties

$$\hat{P}_W^2 = \hat{P}_W$$
$$\hat{P}_W^\dagger = \hat{P}_W$$

- Conversely any operator that satisfies

$$\hat{P}^2 = \hat{P} \quad \text{Idempotent}$$
$$\hat{P}^\dagger = \hat{P} \quad \text{Self Adjoint}$$

is a projection operator onto the subspace

$$\mathscr{H}_{\hat{P}} := \left\{ \vec{\psi} \in \mathscr{H} \mid \hat{P}\vec{\psi} = \vec{\psi} \right\}$$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-2

- Projection operator properties

$$\hat{\mathrm{P}}_W^2 = \hat{\mathrm{P}}_W$$
$$\hat{\mathrm{P}}_W^\dagger = \hat{\mathrm{P}}_W$$

- Conversely any operator that satisfies

$$\hat{\mathrm{P}}^2 = \hat{\mathrm{P}} \text{ Idempotent}$$
$$\hat{\mathrm{P}}^\dagger = \hat{\mathrm{P}} \text{ Self Adjoint}$$

is a projection operator onto the subspace

$$\mathscr{H}_{\hat{\mathrm{P}}} := \left\{ \vec{\psi} \in \mathscr{H} \mid \hat{\mathrm{P}}\vec{\psi} = \vec{\psi} \right\}$$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-2

- Projection operator properties

$$\hat{\mathrm{P}}_W^2 = \hat{\mathrm{P}}_W$$
$$\hat{\mathrm{P}}_W^\dagger = \hat{\mathrm{P}}_W$$

- Conversely any operator that satisfies

$$\hat{\mathrm{P}}^2 = \hat{\mathrm{P}} \ \text{ Idempotent}$$
$$\hat{\mathrm{P}}^\dagger = \hat{\mathrm{P}} \ \text{ Self Adjoint}$$

is a projection operator onto the subspace

$$\mathscr{H}_{\hat{\mathrm{P}}} := \left\{ \vec{\psi} \in \mathscr{H} \mid \hat{\mathrm{P}}\vec{\psi} = \vec{\psi} \right\}$$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

- Projection operators $\hat{P}_1$ and $\hat{P}_2$ are called orthogonal if

$$\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1 = 0.$$

- If $\hat{P}_1$ and $\hat{P}_2$ are orthogonal,

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

- Projection operators $\hat{P}_1$ and $\hat{P}_2$ are called orthogonal if

$$\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1 = 0.$$

- If $\hat{P}_1$ and $\hat{P}_2$ are orthogonal, then $\hat{P}_1 + \hat{P}_2$ is also a projection operator.
- If $|e_1\rangle, |e_2\rangle, \ldots$ is an orthonormal basis for $\mathscr{H}$ then

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

- Projection operators $\hat{\mathrm{P}}_1$ and $\hat{\mathrm{P}}_2$ are called orthogonal if

$$\hat{\mathrm{P}}_1\hat{\mathrm{P}}_2 = \hat{\mathrm{P}}_2\hat{\mathrm{P}}_1 = 0.$$

- If $\hat{\mathrm{P}}_1$ and $\hat{\mathrm{P}}_2$ are orthogonal, then $\hat{\mathrm{P}}_1 + \hat{\mathrm{P}}_2$ is also a projection operator.
- If $|e_1\rangle, |e_2\rangle, \ldots$ is an orthonormal basis for $\mathscr{H}$ then

$$|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, \ldots |e_i\rangle\langle e_i|, \ldots$$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

- Projection operators $\hat{\mathrm{P}}_1$ and $\hat{\mathrm{P}}_2$ are called orthogonal if

$$\hat{\mathrm{P}}_1\hat{\mathrm{P}}_2 = \hat{\mathrm{P}}_2\hat{\mathrm{P}}_1 = 0.$$

- If $\hat{\mathrm{P}}_1$ and $\hat{\mathrm{P}}_2$ are orthogonal, then $\hat{\mathrm{P}}_1 + \hat{\mathrm{P}}_2$ is also a projection operator.
- If $|e_1\rangle, |e_2\rangle, \ldots$ is an orthonormal basis for $\mathscr{H}$ then

$$|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, \ldots |e_i\rangle\langle e_i|, \ldots$$

are projection operators onto

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

- Projection operators $\hat{P}_1$ and $\hat{P}_2$ are called orthogonal if

$$\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1 = 0.$$

- If $\hat{P}_1$ and $\hat{P}_2$ are orthogonal, then $\hat{P}_1 + \hat{P}_2$ is also a projection operator.
- If $|e_1\rangle, |e_2\rangle, \ldots$ is an orthonormal basis for $\mathscr{H}$ then

$$|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, \ldots |e_i\rangle\langle e_i|, \ldots$$

  are projection operators onto the one dimensional subspaces $|e_1\rangle, |e_2\rangle, \ldots, |e_i\rangle, \ldots$ respectively.
- Then $|e_i\rangle\langle e_i|, |e_i\rangle\langle e_i|$ are orthogonal if $i \neq j$.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-3

- Projection operators $\hat{\mathrm{P}}_1$ and $\hat{\mathrm{P}}_2$ are called orthogonal if

$$\hat{\mathrm{P}}_1\hat{\mathrm{P}}_2 = \hat{\mathrm{P}}_2\hat{\mathrm{P}}_1 = 0.$$

- If $\hat{\mathrm{P}}_1$ and $\hat{\mathrm{P}}_2$ are orthogonal, then $\hat{\mathrm{P}}_1 + \hat{\mathrm{P}}_2$ is also a projection operator.
- If $|e_1\rangle, |e_2\rangle, \ldots$ is an orthonormal basis for $\mathscr{H}$ then

$$|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, \ldots |e_i\rangle\langle e_i|, \ldots$$

  are projection operators onto the one dimensional subspaces $|e_1\rangle, |e_2\rangle, \ldots, |e_i\rangle, \ldots$ respectively.
- Then $|e_i\rangle\langle e_i|, |e_i\rangle\langle e_i|$ are orthogonal if $i \neq j$.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

- A Hilbert space is the direct sum of a collection, $W_1, W_2, \ldots, W_M$, of closed subspaces (M could be infinite) if

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

- A Hilbert space is the direct sum of a collection, $W_1, W_2, \ldots, W_M$, of closed subspaces (M could be infinite) if (i) these subspaces are mutually orthogonal, and (ii) every vector in the Hilbert space is equal to

$$\vec{\psi} = \hat{P}_{W_1}\vec{\psi} + \hat{P}_{W_2}\vec{\psi} + \ldots + \hat{P}_{W_M}\vec{\psi}$$

Obviously,

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

- A Hilbert space is the direct sum of a collection, $W_1, W_2, \ldots, W_M$, of closed subspaces (M could be infinite) if (i) these subspaces are mutually orthogonal, and (ii) every vector in the Hilbert space is equal to

$$\vec{\psi} = \hat{P}_{W_1}\vec{\psi} + \hat{P}_{W_2}\vec{\psi} + \ldots + \hat{P}_{W_M}\vec{\psi}$$

Obviously, $\hat{P}_{W_i}\hat{P}_{W_j} = \delta_{ij}\hat{P}_{W_i}$ and

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

- A Hilbert space is the direct sum of a collection, $W_1, W_2, \ldots, W_M$, of closed subspaces (M could be infinite) if (i) these subspaces are mutually orthogonal, and (ii) every vector in the Hilbert space is equal to

$$\vec{\psi} = \hat{P}_{W_1}\vec{\psi} + \hat{P}_{W_2}\vec{\psi} + \ldots + \hat{P}_{W_M}\vec{\psi}$$

Obviously, $\hat{P}_{W_i}\hat{P}_{W_j} = \delta_{ij}\hat{P}_{W_i}$ and

$$\hat{P}_{W_1} + \hat{P}_{W_2} + \cdots + \hat{P}_{W_M} = \hat{\mathbb{1}}.$$

The last equation is called the

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

- A Hilbert space is the direct sum of a collection, $W_1, W_2, \ldots, W_M$, of closed subspaces (M could be infinite) if (i) these subspaces are mutually orthogonal, and (ii) every vector in the Hilbert space is equal to

$$\vec{\psi} = \hat{P}_{W_1}\vec{\psi} + \hat{P}_{W_2}\vec{\psi} + \ldots + \hat{P}_{W_M}\vec{\psi}$$

Obviously, $\hat{P}_{W_i}\hat{P}_{W_j} = \delta_{ij}\hat{P}_{W_i}$ and

$$\hat{P}_{W_1} + \hat{P}_{W_2} + \cdots + \hat{P}_{W_M} = \hat{\mathbb{1}}.$$

The last equation is called the resolution of the identity.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators-4

- A Hilbert space is the direct sum of a collection, $W_1, W_2, \ldots, W_M$, of closed subspaces (M could be infinite) if (i) these subspaces are mutually orthogonal, and (ii) every vector in the Hilbert space is equal to

$$\vec{\psi} = \hat{P}_{W_1}\vec{\psi} + \hat{P}_{W_2}\vec{\psi} + \ldots + \hat{P}_{W_M}\vec{\psi}$$

Obviously, $\hat{P}_{W_i}\hat{P}_{W_j} = \delta_{ij}\hat{P}_{W_i}$ and

$$\hat{P}_{W_1} + \hat{P}_{W_2} + \cdots + \hat{P}_{W_M} = \hat{\mathbb{1}}.$$

The last equation is called the resolution of the identity.

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \, ,$$

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \, ,$$

$d(m) =$ degree of degeneracy of eigenvalue m.

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \,,$$

$d(m) =$ degree of degeneracy of eigenvalue m.

- Of course

$$\hat{\mathbb{1}} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j| \equiv \sum_{m=1}^{M} \hat{P}_m$$

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \,,$$

$d(m) =$ degree of degeneracy of eigenvalue m.

- Of course

$$\hat{\mathbb{1}} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j| \equiv \sum_{m=1}^{M} \hat{P}_m$$

where $\hat{P}_m \equiv \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j|$:

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \, ,$$

$d(m) =$ degree of degeneracy of eigenvalue m.

- Of course

$$\hat{\mathbb{1}} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j| \equiv \sum_{m=1}^{M} \hat{P}_m$$

where $\hat{P}_m \equiv \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j|$: Spectral Projector:

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \, ,$$

  $d(m) =$ degree of degeneracy of eigenvalue m.

- Of course

$$\hat{\mathbb{1}} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j| \equiv \sum_{m=1}^{M} \hat{P}_m$$

  where $\hat{P}_m \equiv \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j|$: Spectral Projector: i.e. projector to subspace with eigenvalue a_m.

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-1

- The set of all eigenvectors of a self adjoint operator forms and orthonormal basis for the Hilbert space $\mathscr{H}$, i.e.,

$$|\psi\rangle = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} \langle a_m, j|\psi\rangle \, |a_m, j\rangle \, ,$$

$d(m) =$ degree of degeneracy of eigenvalue m.

- Of course

$$\hat{\mathbb{1}} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j| \equiv \sum_{m=1}^{M} \hat{P}_m$$

where $\hat{P}_m \equiv \sum_{j=1}^{d(m)} |a_m, j\rangle \langle a_m, j|$: Spectral Projector: i.e. projector to subspace with eigenvalue a_m.

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-2

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-2

- Then

$$\hat{A} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} a_m \left|a_m, j\right\rangle \left\langle a_m, j\right| \equiv \sum_{m=1}^{M} a_m \hat{P}_m$$

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-2

- Then

$$\hat{A} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} a_m \left|a_m, j\right\rangle \left\langle a_m, j\right| \equiv \sum_{m=1}^{M} a_m \hat{P}_m$$

Spectral Representation of $\hat{A}$

The Kochen-Specker Theorem

Mathematical Digression: Self Adjoint Operators-2

- Then

$$\hat{A} = \sum_{m=1}^{M} \sum_{j=1}^{d(m)} a_m \left|a_m, j\right\rangle \left\langle a_m, j\right| \equiv \sum_{m=1}^{M} a_m \hat{P}_m$$

Spectral Representation of $\hat{A}$

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues:

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\{W^{\perp}\} \geq 1$
- because of their eigenvalues,

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\{W^{\perp}\} \geq 1$
- because of their eigenvalues, they correspond to "binary-valued" observables

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\left\{W^{\perp}\right\} \geq 1$
- because of their eigenvalues, they correspond to "binary-valued" observables and can, thus, be understood as propositions about properties of the quantum system.

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\{W^{\perp}\} \geq 1$
- because of their eigenvalues, they correspond to "binary-valued" observables and can, thus, be understood as propositions about properties of the quantum system. Very Important for Conceptual foundations!

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\{W^{\perp}\} \geq 1$
- because of their eigenvalues, they correspond to "binary-valued" observables and can, thus, be understood as propositions about properties of the quantum system. Very Important for Conceptual foundations!
- Commuting operators correspond to "comeasurable observables", thus

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\left\{W^{\perp}\right\} \geq 1$
- because of their eigenvalues, they correspond to "binary-valued" observables and can, thus, be understood as propositions about properties of the quantum system. Very Important for Conceptual foundations!
- Commuting operators correspond to "comeasurable observables", thus commuting projection operators "correspond to simultaneous propositions".

The Kochen-Specker Theorem

Mathematical Digression: Projection Operators, Significance

- Non-trivial projection operators have just two eigenvalues: 1 and 0. Both degenerate if $\dim\{W\} \geq 1$ and $\dim\{W^{\perp}\} \geq 1$
- because of their eigenvalues, they correspond to ”binary-valued” observables and can, thus, be understood as propositions about properties of the quantum system. Very Important for Conceptual foundations!
- Commuting operators correspond to ”comeasurable observables”, thus commuting projection operators ”correspond to simultaneous propositions”.

The Kochen-Specker Theorem

Core of Mathematical Argument.

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra.

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra. But every projection corresponds uniquely to a closed linear subspace

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra. But every projection corresponds uniquely to a closed linear subspace we may thus consider $\mathscr{P}(\mathscr{H})$ as the partial Boolean algebra of closed linear subspaces of $\mathscr{H}$.

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra. But every projection corresponds uniquely to a closed linear subspace we may thus consider $\mathscr{P}(\mathscr{H})$ as the partial Boolean algebra of closed linear subspaces of $\mathscr{H}$.
- Then, existence of hidden variables

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra. But every projection corresponds uniquely to a closed linear subspace we may thus consider $\mathscr{P}(\mathscr{H})$ as the partial Boolean algebra of closed linear subspaces of $\mathscr{H}$.
- Then, existence of hidden variables implies the existence of an imbedding of the partial Boolean algebra of propositions of QM into a Boolean algebra.

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra. But every projection corresponds uniquely to a closed linear subspace we may thus consider $\mathscr{P}(\mathscr{H})$ as the partial Boolean algebra of closed linear subspaces of $\mathscr{H}$.
- Then, existence of hidden variables implies the existence of an imbedding of the partial Boolean algebra of propositions of QM into a Boolean algebra.

The Kochen-Specker Theorem

Core of Mathematical Argument.

- The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute.
- Then, a necessary condition for the existence of hidden variables is that this partial algebra be imbeddable in a commutative algebra
- The set $\mathscr{P}(\mathscr{H})$ of projections forms a partial Boolean algebra. But every projection corresponds uniquely to a closed linear subspace we may thus consider $\mathscr{P}(\mathscr{H})$ as the partial Boolean algebra of closed linear subspaces of $\mathscr{H}$.
- Then, existence of hidden variables implies the existence of an imbedding of the partial Boolean algebra of propositions of QM into a Boolean algebra.

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(\mathcal{Q})$ for the value of the physical quantity $\mathcal{Q}$ in the quantum state $\vec{\psi}$ of an individual system exists.

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(\mathcal{Q})$ for the value of the physical quantity $\mathcal{Q}$ in the quantum state $\vec{\psi}$ of an individual system exists. We require:

1. $$V_{\vec{\psi}}\left(\mathscr{F}(\mathcal{Q})\right) = \mathscr{F}\left(V_{\vec{\psi}}(\mathcal{Q})\right), \ \mathscr{F}: \mathbb{R} \to \mathbb{R}$$

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

1.
$$V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right), \ \mathscr{F}:\mathbb{R}\to\mathbb{R} \qquad [1]$$

I.e.,

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

1.
$$V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right), \ \mathscr{F}:\mathbb{R}\to\mathbb{R} \qquad [1]$$

I.e., in any quantum state the value of a function of Q is equal to the function of the value of Q.

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

1.
$$V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right), \ \mathscr{F}: \mathbb{R} \to \mathbb{R} \qquad [1]$$

   I.e., in any quantum state the value of a function of Q is equal to the function of the value of Q. Of Fundmanental Importance!

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(\mathcal{Q})$ for the value of the physical quantity $\mathcal{Q}$ in the quantum state $\vec{\psi}$ of an individual system exists. We require:

  1. $$V_{\vec{\psi}}\left(\mathscr{F}(\mathcal{Q})\right) = \mathscr{F}\left(V_{\vec{\psi}}(\mathcal{Q})\right), \ \mathscr{F} : \mathbb{R} \to \mathbb{R} \qquad [1]$$

     I.e., in any quantum state the value of a function of $\mathcal{Q}$ is equal to the function of the value of $\mathcal{Q}$. Of Fundmanental Importance!

  - We may think of $V_{\vec{\psi}}$ as a function of self-adjoint operators

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

1. $$V_{\vec{\psi}}(\mathscr{F}(Q)) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right), \ \mathscr{F} : \mathbb{R} \to \mathbb{R} \qquad [1]$$

   I.e., in any quantum state the value of a function of Q is equal to the function of the value of Q. Of Fundmanental Importance!

   - We may think of $V_{\vec{\psi}}$ as a function of self-adjoint operators (under quantization maps that are onto and one-to-one).

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

  1. $$V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right), \; \mathscr{F}: \mathbb{R} \to \mathbb{R} \qquad [1]$$

     I.e., in any quantum state the value of a function of Q is equal to the function of the value of Q. Of Fundmanental Importance!

     - We may think of $V_{\vec{\psi}}$ as a function of self-adjoint operators (under quantization maps that are onto and one-to-one).

The Kochen-Specker Theorem (Kochen and Specker 1967)

Preparation.

- The attempted imbedding will be what we shall call Value Function.
- Suppose, a value function $V_{\vec{\psi}}(Q)$ for the value of the physical quantity Q in the quantum state $\vec{\psi}$ of an individual system exists. We require:

1. $$V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right),\ \mathscr{F} : \mathbb{R} \to \mathbb{R} \qquad [1]$$

   I.e., in any quantum state the value of a function of Q is equal to the function of the value of Q. Of Fundmanental Importance!

   - We may think of $V_{\vec{\psi}}$ as a function of self-adjoint operators (under quantization maps that are onto and one-to-one).

The Kochen-Specker Theorem

Implications of [1]

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A + B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right),$$

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A + B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right),$$ Additivity

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A + B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right), \text{ Additivity} \qquad [2]$$

$A + B$: Physical Quantity (PQ) that corresponds to $\hat{A} + \hat{B}$

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A+B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right),$$ Additivity [2]

$A+B$: Physical Quantity (PQ) that corresponds to $\hat{A}+\hat{B}$

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(AB\right) = V_{\vec{\psi}}\left(A\right) V_{\vec{\psi}}\left(B\right)$$

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A + B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right), \text{ Additivity} \qquad [2]$$

  $A + B$: Physical Quantity (PQ) that corresponds to $\hat{A} + \hat{B}$
- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(AB\right) = V_{\vec{\psi}}\left(A\right) V_{\vec{\psi}}\left(B\right) \text{ Mulitplicativity}$$

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A+B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right),$$ Additivity [2]

$A + B$: Physical Quantity (PQ) that corresponds to $\hat{A} + \hat{B}$

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(AB\right) = V_{\vec{\psi}}\left(A\right) V_{\vec{\psi}}\left(B\right)$$ Mulitplicativity [3]

AB: PQ that corresponds to $\hat{A}\hat{B}$.

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A + B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right),\ \text{Additivity} \qquad [2]$$

  $A + B$: Physical Quantity (PQ) that corresponds to $\hat{A} + \hat{B}$
- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(AB\right) = V_{\vec{\psi}}\left(A\right) V_{\vec{\psi}}\left(B\right) \text{Mulitplicativity} \qquad [3]$$

  AB: PQ that corresponds to $\hat{A}\hat{B}$.
- If $\mathbb{1}$ is the PQ that corresponds to $\hat{\mathbb{1}}$ then

$$V_{\vec{\psi}}\left(\mathbb{1}\right) = 1 \qquad [4]$$

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A + B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right), \text{ Additivity} \qquad [2]$$

  $A + B$: Physical Quantity (PQ) that corresponds to $\hat{A} + \hat{B}$

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(AB\right) = V_{\vec{\psi}}\left(A\right) V_{\vec{\psi}}\left(B\right) \text{ Mulitplicativity} \qquad [3]$$

  AB: PQ that corresponds to $\hat{A}\hat{B}$.

- If $\mathbb{1}$ is the PQ that corresponds to $\hat{\mathbb{1}}$ then

$$V_{\vec{\psi}}\left(\mathbb{1}\right) = 1 \qquad [4]$$

  if there is at least one PQ for which $V_{\vec{\psi}}\left(Q\right) \neq 0$

The Kochen-Specker Theorem

Implications of [1]

- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(A+B\right) = V_{\vec{\psi}}\left(A\right) + V_{\vec{\psi}}\left(B\right), \text{ Additivity} \qquad [2]$$

  $A+B$: Physical Quantity (PQ) that corresponds to $\hat{A}+\hat{B}$
- If $\left[\hat{A},\ \hat{B}\right] = 0$, then

$$V_{\vec{\psi}}\left(AB\right) = V_{\vec{\psi}}\left(A\right) V_{\vec{\psi}}\left(B\right) \text{ Mulitplicativity} \qquad [3]$$

  AB: PQ that corresponds to $\hat{A}\hat{B}$.
- If $\mathbb{1}$ is the PQ that corresponds to $\hat{\mathbb{1}}$ then

$$V_{\vec{\psi}}\left(\mathbb{1}\right) = 1 \qquad [4]$$

  if there is at least one PQ for which $V_{\vec{\psi}}\left(Q\right) \neq 0$

The Kochen-Specker Theorem

Implications of [3]

The Kochen-Specker Theorem

Implications of [3]

- Suppose P is the Physical Quantity that corresponds to the projection operator $\hat{P}$ then

$$V_{\vec{\psi}}\left(P\right) = 0 \text{ or } 1.$$

The Kochen-Specker Theorem

Implications of [3]

- Suppose P is the Physical Quantity that corresponds to the projection operator $\widehat{P}$ then

$$V_{\vec{\psi}}\left(P\right)=0 \text{ or } 1. \qquad [5]$$

But we think of P as a proposition, thus a quantum value function $V_{\vec{\psi}}$ gives it a "false" or "true" ascription in the state $\vec{\psi}$.

The Kochen-Specker Theorem

Implications of [3]

- Suppose P is the Physical Quantity that corresponds to the projection operator $\hat{P}$ then

$$V_{\vec{\psi}}\,(P) = 0 \text{ or } 1. \qquad [5]$$

But we think of P as a proposition, thus a quantum value function $V_{\vec{\psi}}$ gives it a "false" or "true" ascription in the state $\vec{\psi}$.

The Kochen-Specker Theorem

The Argument

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example,

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues,

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \dots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \dots \hat{P}_n\right\}$ should be
  - mutually exclusive:

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be
    - mutually exclusive: only can be true at a given time

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive:

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive: at least one of them must be true at any given time.

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \dots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \dots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive: at least one of them must be true at any given time.

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive: at least one of them must be true at any given time.

In a set of mutually exclusive and exhaustive propositions

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive: at least one of them must be true at any given time.

  In a set of mutually exclusive and exhaustive propositions one-and only one-can be true. The rest are false.
- In view of the above requirement

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive: at least one of them must be true at any given time.

  In a set of mutually exclusive and exhaustive propositions one-and only one-can be true. The rest are false.
- In view of the above requirement we take projections $\hat{P}_i := |e_i\rangle\langle e_i|$ to the one-dimensional subspaces formed by the basis elements of an orthonormal basis set of $\mathscr{H}$.

The Kochen-Specker Theorem

The Argument

- Suppose there is a collection $\left\{\hat{P}_1, \hat{P}_2, \dots \hat{P}_n\right\}$ of projection operators that form a resolution of the identity.
- Example, spectral projectors of a self-adjoint operator $\hat{A}$ with discrete eigenvalues, i.e. $\hat{P}_i = \hat{P}_{A=a_i}$.
- If a value of the quantity A exists then the collection of propositions $\left\{\hat{P}_1, \hat{P}_2, \dots \hat{P}_n\right\}$ should be
  - mutually exclusive: only can be true at a given time and
  - exhaustive: at least one of them must be true at any given time.

  In a set of mutually exclusive and exhaustive propositions one-and only one-can be true. The rest are false.
- In view of the above requirement we take projections $\hat{P}_i := |e_i\rangle\langle e_i|$ to the one-dimensional subspaces formed by the basis elements of an orthonormal basis set of $\mathscr{H}$.

The Kochen-Specker Theorem

Argument's Core

The Kochen-Specker Theorem

Argument's Core

The Kochen-Specker Theorem

Argument's Core

- Then, for any state $\vec{\psi}$ a value function $V_{\vec{\psi}}$ must assign the number 1 to one and only one of the projectors/one-dimensional subspaces and O to the rest.

The Kochen-Specker Theorem

Argument's Core

- Then, for any state $\vec{\psi}$ a value function $V_{\vec{\psi}}$ must assign the number 1 to one and only one of the projectors/one-dimensional subspaces and O to the rest.

Statement:

The Kochen-Specker Theorem

Argument's Core

- Then, for any state $\vec{\psi}$ a value function $V_{\vec{\psi}}$ must assign the number 1 to one and only one of the projectors/one-dimensional subspaces and O to the rest.

Statement:

Let H be a Hilbert Space of dimension $d \geq 3$, it is impossible to associate definite numerical values 1 or 0, with every projection operator $\hat{\mathrm{P}}_m$, in such a way that, if a set of commuting $\hat{\mathrm{P}}_m$ satisfies $\sum \hat{\mathrm{P}}_m = \hat{\mathbb{1}}$, the corresponding values, namely $V(\hat{\mathrm{P}}_m) = 0$ or 1, also satisfy $\sum V(\hat{\mathrm{P}}_m) = 1$

The Kochen-Specker Theorem

Argument's Core

- Then, for any state $\vec{\psi}$ a value function $V_{\vec{\psi}}$ must assign the number 1 to one and only one of the projectors/one-dimensional subspaces and O to the rest.

Statement:

Let H be a Hilbert Space of dimension $d \geq 3$, it is impossible to associate definite numerical values 1 or 0, with every projection operator $\hat{\mathrm{P}}_m$, in such a way that, if a set of commuting $\hat{\mathrm{P}}_m$ satisfies $\sum \hat{\mathrm{P}}_m = \hat{\mathbb{1}}$, the corresponding values, namely $V(\hat{\mathrm{P}}_m) = 0$ or 1, also satisfy $\sum V(\hat{\mathrm{P}}_m) = 1$

The Kochen-Specker Theorem

Steps of the Proof

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n,

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$ suffices to show that no such function can exist in real, 3-dimensional euclidean space $\mathbb{R}^3$.

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$ suffices to show that no such function can exist in real, 3-dimensional euclidean space $\mathbb{R}^3$.
4. Assume that if, in $\mathbb{R}^3$, a vector is a member of more than one basis,

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$ suffices to show that no such function can exist in real, 3-dimensional euclidean space $\mathbb{R}^3$.
4. Assume that if, in $\mathbb{R}^3$, a vector is a member of more than one basis, the value 1 or 0 associated with that vector is the same,

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$ suffices to show that no such function can exist in real, 3-dimensional euclidean space $\mathbb{R}^3$.
4. Assume that if, in $\mathbb{R}^3$, a vector is a member of more than one basis, the value 1 or 0 associated with that vector is the same, irrespective of the choice of the other basis vectors.

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$ suffices to show that no such function can exist in real, 3-dimensional euclidean space $\mathbb{R}^3$.
4. Assume that if, in $\mathbb{R}^3$, a vector is a member of more than one basis, the value 1 or 0 associated with that vector is the same, irrespective of the choice of the other basis vectors.
5. Kochen and Specker found a counter example, with the use of a set of 117 vectors in the vector space $\mathbb{R}^3$

The Kochen-Specker Theorem

Steps of the Proof

1. If a value function exists for a Hilbert space of a particular dimension n, then it necessarily does so for any space of dimension less than n.
2. If a value function exists for a complex Hilbert space, then one also exists on any real Hilbert space of the same dimension.
3. $(1) + (2) \Rightarrow$ suffices to show that no such function can exist in real, 3-dimensional euclidean space $\mathbb{R}^3$.
4. Assume that if, in $\mathbb{R}^3$, a vector is a member of more than one basis, the value 1 or 0 associated with that vector is the same, irrespective of the choice of the other basis vectors.
5. Kochen and Specker found a counter example, with the use of a set of 117 vectors in the vector space $\mathbb{R}^3$

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges.

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$.

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism:

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $\left(\bar{1} \to -1\right)$, $\left(2 \to \sqrt{2}\right)$, and $\left(\bar{2} \to -\sqrt{2}\right)$.

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $(\bar{1} \to -1)$, $\left(2 \to \sqrt{2}\right)$, and $\left(\bar{2} \to -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once:

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $(\bar{1} \to -1)$, $\left(2 \to \sqrt{2}\right)$, and $\left(\bar{2} \to -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once: they correspond to the same projector.

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $(\bar{1} \rightarrow -1)$, $\left(2 \rightarrow \sqrt{2}\right)$, and $\left(\bar{2} \rightarrow -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once: they correspond to the same projector. Rays associated with the values 1 and 0 will be called blue and red, respectively.

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $(\bar{1} \to -1)$, $\left(2 \to \sqrt{2}\right)$, and $\left(\bar{2} \to -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once: they correspond to the same projector. Rays associated with the values 1 and 0 will be called blue and red, respectively. The set of rays is invariant under interchanges of the x, y, and z axes, and under a reversal of the direction of each axis.

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $\left(\bar{1} \to -1\right)$, $\left(2 \to \sqrt{2}\right)$, and $\left(\bar{2} \to -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once: they correspond to the same projector. Rays associated with the values 1 and 0 will be called blue and red, respectively. The set of rays is invariant under interchanges of the x, y, and z axes, and under a reversal of the direction of each axis. This allows us to

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism: 0, 1, $\left(\bar{1} \to -1\right)$, $\left(2 \to \sqrt{2}\right)$, and $\left(\bar{2} \to -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once: they correspond to the same projector. Rays associated with the values 1 and 0 will be called blue and red, respectively. The set of rays is invariant under interchanges of the x, y, and z axes, and under a reversal of the direction of each axis. This allows us to assign arbitrarily—without loss of generality—the value 1 to some of the rays

The Kochen-Specker Theorem

Asher Peres' Proof (Peres 1991)

Asher Peres proved the theorem using 33 rays (only directions are relevant). These are obtained by connecting the center of a cube which serves as the origin, to "suitably chosen" points on its faces and edges. He labelled each with (x, y, z) where x, y, and z are the coordinates of the point. These can be: 0, 1, -1, $\sqrt{2}$, and $-\sqrt{2}$. For convenience he chose the following symbolism:
0, 1, $\left(\bar{1} \rightarrow -1\right)$, $\left(2 \rightarrow \sqrt{2}\right)$, and $\left(\bar{2} \rightarrow -\sqrt{2}\right)$.

Opposite rays, such as $\bar{2}01$ and $20\bar{1}$, are counted only once: they correspond to the same projector. Rays associated with the values 1 and 0 will be called blue and red, respectively. The set of rays is invariant under interchanges of the x, y, and z axes, and under a reversal of the direction of each axis. This allows us to assign arbitrarily—without loss of generality—the value 1 to some of the rays

The Kochen-Specker Theorem

Asher Peres’ Proof

The Kochen-Specker Theorem

Asher Peres’ Proof

Only Rays need are present in table.

The Kochen-Specker Theorem

Asher Peres' Proof

Only Rays need are present in table.

Orthogonal Triad			Other rays		Why 1st Ray is Blue
001	100	010	110	$1\bar{1}0$	arbitrary choice of z axis
101	$\bar{1}01$	010			arbitrary choice of x vs $-x$
011	$0\bar{1}1$	100			arbitrary choice of y vs $-y$
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021	arbitrary choice of x vs y
102	$\bar{2}01$	010	$\bar{2}11$		orthogonality to 2nd and 3rd rays
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$		orthogonality to 2nd and 3rd rays
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$		orthogonality to 2nd and 3rd rays
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$		orthogonality to 2nd and 3rd rays
012	100	$0\bar{2}1$	$1\bar{2}1$		orthogonality to 2nd and 3rd rays
121	$\bar{1}01$	$1\bar{2}1$	$01\bar{2}$		orthogonality to 2nd and 3rd rays

The Kochen-Specker Theorem

Asher Peres' Proof

Only Rays need are present in table.

Orthogonal Triad			Other rays		Why 1st Ray is Blue
001	100	010	110	$1\bar{1}0$	arbitrary choice of z axis
101	$\bar{1}01$	010			arbitrary choice of x vs $-x$
011	$0\bar{1}1$	100			arbitrary choice of y vs $-y$
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021	arbitrary choice of x vs y
102	$\bar{2}01$	010	$\bar{2}11$		orthogonality to 2nd and 3rd rays
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$		orthogonality to 2nd and 3rd rays
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$		orthogonality to 2nd and 3rd rays
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$		orthogonality to 2nd and 3rd rays
012	100	$0\bar{2}1$	$1\bar{2}1$		orthogonality to 2nd and 3rd rays
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$		orthogonality to 2nd and 3rd rays

Now Observe first fourth and last lines.

The Kochen-Specker Theorem

Asher Peres' Proof

Only Rays need are present in table.

Orthogonal Triad			Other rays		Why 1st Ray is Blue
001	100	010	110	$1\bar{1}0$	arbitrary choice of z axis
101	$\bar{1}01$	010			arbitrary choice of x vs $-x$
011	$0\bar{1}1$	100			arbitrary choice of y vs $-y$
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021	arbitrary choice of x vs y
102	$\bar{2}01$	010	$\bar{2}11$		orthogonality to 2nd and 3rd rays
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$		orthogonality to 2nd and 3rd rays
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$		orthogonality to 2nd and 3rd rays
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$		orthogonality to 2nd and 3rd rays
012	100	$0\bar{2}1$	$1\bar{2}1$		orthogonality to 2nd and 3rd rays
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$		orthogonality to 2nd and 3rd rays

Now Observe first fourth and last lines.

The Kochen-Specker Theorem

Asher Peres' Proof

Only Rays need are present in table.

Orthogonal Triad			Other rays		Why 1st Ray is Blue
001	100	010	110	$1\bar{1}0$	arbitrary choice of z axis
101	$\bar{1}01$	010			arbitrary choice of x vs $-x$
011	$0\bar{1}1$	100			arbitrary choice of y vs $-y$
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021	arbitrary choice of x vs y
102	$\bar{2}01$	010	$\bar{2}11$		orthogonality to 2nd and 3rd rays
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$		orthogonality to 2nd and 3rd rays
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$		orthogonality to 2nd and 3rd rays
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$		orthogonality to 2nd and 3rd rays
012	100	$0\bar{2}1$	$1\bar{2}1$		orthogonality to 2nd and 3rd rays
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$		orthogonality to 2nd and 3rd rays

Now Observe first fourth and last lines.

The Kochen-Specker Theorem

Proof continues

The Kochen-Specker Theorem

Proof continues

Orthogonal Triad			Other rays	
001	100	010	110	$1\bar{1}0$
101	$\bar{1}01$	010		
011	$0\bar{1}1$	100		
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021
102	$\bar{2}01$	010	$\bar{2}11$	
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$	
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$	
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$	
012	100	$0\bar{2}1$	$1\bar{2}1$	
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$	

The Kochen-Specker Theorem

Proof continues

Orthogonal Triad			Other rays	
001	100	010	110	$1\bar{1}0$
101	$\bar{1}01$	010		
011	$0\bar{1}1$	100		
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021
102	$\bar{2}01$	010	$\bar{2}11$	
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$	
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$	
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$	
012	100	$0\bar{2}1$	$1\bar{2}1$	
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$	

The elements 100, 021, and $0\bar{1}2$:

The Kochen-Specker Theorem

Proof continues

Orthogonal Triad			Other rays	
001	100	010	110	$1\bar{1}0$
101	$\bar{1}01$	010		
011	$0\bar{1}1$	100		
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021
102	$\bar{2}01$	010	$\bar{2}11$	
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$	
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$	
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$	
012	100	$0\bar{2}1$	$1\bar{2}1$	
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$	

The elements 100, 021, and $0\bar{1}2$: are red and mutually orhtogonal!

The Kochen-Specker Theorem

Proof continues

Orthogonal Triad			Other rays	
001	100	010	110	$1\bar{1}0$
101	$\bar{1}01$	010		
011	$0\bar{1}1$	100		
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021
102	$\bar{2}01$	010	$\bar{2}11$	
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$	
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$	
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$	
012	100	$0\bar{2}1$	$1\bar{2}1$	
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$	

The elements 100, 021, and $0\bar{1}2$: are red and mutually orhtogonal! Hence Contadicion!

The Kochen-Specker Theorem

Proof continues

Orthogonal Triad			Other rays	
001	100	010	110	$1\bar{1}0$
101	$\bar{1}01$	010		
011	$0\bar{1}1$	100		
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021
102	$\bar{2}01$	010	$\bar{2}11$	
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$	
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$	
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$	
012	100	$0\bar{2}1$	$1\bar{2}1$	
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$	

The elements 100, 021, and $0\bar{1}2$: are red and mutually orhtogonal! Hence Contadicion!

The Kochen-Specker Theorem

Proof continues

Orthogonal Triad			Other rays	
001	100	010	110	$1\bar{1}0$
101	$\bar{1}01$	010		
011	$0\bar{1}1$	100		
$1\bar{1}2$	$\bar{1}12$	110	$\bar{2}01$	021
102	$\bar{2}01$	010	$\bar{2}11$	
211	$0\bar{1}1$	$\bar{2}11$	$\bar{1}02$	
201	010	$\bar{1}02$	$\bar{1}\bar{1}2$	
112	$1\bar{1}0$	$\bar{1}\bar{1}2$	$0\bar{2}1$	
012	100	$0\bar{2}1$	$1\bar{2}1$	
121	$\bar{1}01$	$1\bar{2}1$	$0\bar{1}2$	

The elements 100, 021, and $0\bar{1}2$: are red and mutually orhtogonal! Hence Contadicion!

The Kochen-Specker Theorem

Implications

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(\mathcal{Q})\right) = \mathscr{F}\left(V_{\vec{\psi}}(\mathcal{Q})\right)$. Then the question is:

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?

Well, suppose that for some self-adjoint operator $\hat{A}$ we may have

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}\left(Q\right)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?

Well, suppose that for some self-adjoint operator $\hat{A}$ we may have $\hat{A} = f_1\left(\hat{A}_1\right)$ and $\hat{A} = f_2\left(\hat{A}_2\right)$

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?

Well, suppose that for some self-adjoint operator $\hat{A}$ we may have $\hat{A} = f_1\left(\hat{A}_1\right)$ and $\hat{A} = f_2\left(\hat{A}_2\right)$ for $\hat{A}_1$ and $\hat{A}_2$ self-adjoint with $\left[\hat{A}_1, \hat{A}_2\right] \neq 0$.

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}\left(Q\right)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?

Well, suppose that for some self-adjoint operator $\hat{A}$ we may have $\hat{A} = f_1\left(\hat{A}_1\right)$ and $\hat{A} = f_2\left(\hat{A}_2\right)$ for $\hat{A}_1$ and $\hat{A}_2$ self-adjoint with $\left[\hat{A}_1,\, \hat{A}_2\right] \neq 0$.

This would lead to

$$f_1\left(V_{\vec{\psi}}\left(\hat{A}_1\right)\right) = f_2\left(V_{\vec{\psi}}\left(\hat{A}_2\right)\right)$$

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?
Well, suppose that for some self-adjoint operator $\hat{A}$ we may have $\hat{A} = f_1\left(\hat{A}_1\right)$ and $\hat{A} = f_2\left(\hat{A}_2\right)$ for $\hat{A}_1$ and $\hat{A}_2$ self-adjoint with $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$.
This would lead to

$$f_1\left(V_{\vec{\psi}}\left(\hat{A}_1\right)\right) = f_2\left(V_{\vec{\psi}}\left(\hat{A}_2\right)\right) \qquad [6]$$

which implies that taking the result of a measurement of $\hat{A}_1$ and applying f_1 to it would give the same number as taking the result of a measurement of $\hat{A}_2$ and applying f_2 to it.

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?

Well, suppose that for some self-adjoint operator $\hat{A}$ we may have $\hat{A} = f_1\left(\hat{A}_1\right)$ and $\hat{A} = f_2\left(\hat{A}_2\right)$ for $\hat{A}_1$ and $\hat{A}_2$ self-adjoint with $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$.

This would lead to

$$f_1\left(V_{\vec{\psi}}\left(\hat{A}_1\right)\right) = f_2\left(V_{\vec{\psi}}\left(\hat{A}_2\right)\right) \qquad [6]$$

which implies that taking the result of a measurement of $\hat{A}_1$ and applying f_1 to it would give the same number as taking the result of a measurement of $\hat{A}_2$ and applying f_2 to it.

Not very likely, if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$.

The Kochen-Specker Theorem

Implications

If we want a value function $V_{\vec{\psi}}$ to exist we may try to drop [1]: $V_{\vec{\psi}}\left(\mathscr{F}(Q)\right) = \mathscr{F}\left(V_{\vec{\psi}}(Q)\right)$. Then the question is: how can it fail?
Well, suppose that for some self-adjoint operator $\hat{A}$ we may have $\hat{A} = f_1\left(\hat{A}_1\right)$ and $\hat{A} = f_2\left(\hat{A}_2\right)$ for $\hat{A}_1$ and $\hat{A}_2$ self-adjoint with $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$.
This would lead to

$$f_1\left(V_{\vec{\psi}}\left(\hat{A}_1\right)\right) = f_2\left(V_{\vec{\psi}}\left(\hat{A}_2\right)\right) \quad [6]$$

which implies that taking the result of a measurement of $\hat{A}_1$ and applying f_1 to it would give the same number as taking the result of a measurement of $\hat{A}_2$ and applying f_2 to it.
Not very likely, if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$.

The Kochen-Specker Theorem

Contextuality-1

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$.

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation:

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.

Hence a failure of the value condition Eq. [1] and hence of Eq.[6]) means that

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.
Hence a failure of the value condition Eq. [1] and hence of Eq.[6]) means that the value ascribed to the PQ P depends on the context in which P is taken:

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.
Hence a failure of the value condition Eq. [1] and hence of Eq.[6]) means that the value ascribed to the PQ P depends on the context in which P is taken: if viewed as belonging to the spectral representation of $\hat{A}_1$, it will be given one value; if viewed as belonging to that of $\hat{A}_2$, it will be given another!

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.
Hence a failure of the value condition Eq. [1] and hence of Eq.[6]) means that the value ascribed to the PQ P depends on the context in which P is taken: if viewed as belonging to the spectral representation of $\hat{A}_1$, it will be given one value; if viewed as belonging to that of $\hat{A}_2$, it will be given another!
The value of P depends on what other physical quantities are assigned values at the same time:

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.
Hence a failure of the value condition Eq. [1] and hence of Eq.[6]) means that the value ascribed to the PQ P depends on the context in which P is taken: if viewed as belonging to the spectral representation of $\hat{A}_1$, it will be given one value; if viewed as belonging to that of $\hat{A}_2$, it will be given another!
The value of P depends on what other physical quantities are assigned values at the same time: The value depends on any choice made of operators that commute with $\hat{P}$.

The Kochen-Specker Theorem

Contextuality-1

Suppose now that we drop eq. [6] if $\left[\hat{A}_1,\ \hat{A}_2\right] \neq 0$. Then we are faced with the following peculiar situation: Let $\hat{A}_1$ and $\hat{A}_2$ be a pair of non-commuting, self-adjoint operators whose spectral resolutions contain a common projector $\hat{P}$. Then $\hat{P}$ can be written as one function of $\hat{A}_1$ and as another function of $\hat{A}_2$.
Hence a failure of the value condition Eq. [1] and hence of Eq.[6]) means that the value ascribed to the PQ P depends on the context in which P is taken: if viewed as belonging to the spectral representation of $\hat{A}_1$, it will be given one value; if viewed as belonging to that of $\hat{A}_2$, it will be given another!
The value of P depends on what other physical quantities are assigned values at the same time: The value depends on any choice made of operators that commute with $\hat{P}$.

The Kochen-Specker Theorem

Contextuality-2

The Kochen-Specker Theorem

Contextuality-2

If PQ represented by a self-adjoint operator $\hat{A}$ with degenerate eigenvalues then its values may depend on the context:

The Kochen-Specker Theorem

Contextuality-2

If PQ represented by a self-adjoint operator $\hat{A}$ with degenarate eigenvalues then its values may depend on the context: it may depend on the choice of other physical quantities whose associated operators commute with $\hat{A}$, and which therefore help to label vectors in the degeneracy subspaces of $\hat{A}$.

The Kochen-Specker theorem has not directly relevant consequences for the pragmatic approach to quantum theory:

The Kochen-Specker Theorem

Contextuality-2

If PQ represented by a self-adjoint operator $\hat{A}$ with degenarate eigenvalues then its values may depend on the context: it may depend on the choice of other physical quantities whose associated operators commute with $\hat{A}$, and which therefore help to label vectors in the degeneracy subspaces of $\hat{A}$.

The Kochen-Specker theorem has not directly relevant consequences for the pragmatic approach to quantum theory: in this approach, no attempt is made in to think of an individual system as possessing values for its observables.

The Kochen-Specker Theorem

Contextuality-2

If PQ represented by a self-adjoint operator $\hat{A}$ with degenarate eigenvalues then its values may depend on the context: it may depend on the choice of other physical quantities whose associated operators commute with $\hat{A}$, and which therefore help to label vectors in the degeneracy subspaces of $\hat{A}$.

The Kochen-Specker theorem has not directly relevant consequences for the pragmatic approach to quantum theory: in this approach, no attempt is made in to think of an individual system as possessing values for its observables.

The Kochen-Specker Theorem

Contextuality-3

The Kochen-Specker Theorem

Contextuality-3

However, if we insist on the assumptions that (i)

The Kochen-Specker Theorem

Contextuality-3

However, if we insist on the assumptions that (i) physical quantities do have values in any individual system,

The Kochen-Specker Theorem

Contextuality-3

However, if we insist on the assumptions that (i) physical quantities do have values in any individual system, and (ii) that these values can be revealed by suitable measurements,

The Kochen-Specker Theorem

Contextuality-3

However, if we insist on the assumptions that (i) physical quantities do have values in any individual system, and (ii) that these values can be revealed by suitable measurements, then the theorem implies that the results obtained will depend on the context in which an observable is studied.

(Standard) Quantum Mechanics in NON-CONTEXTUAL.

The Kochen-Specker Theorem

Contextuality-3

However, if we insist on the assumptions that (i) physical quantities do have values in any individual system, and (ii) that these values can be revealed by suitable measurements, then the theorem implies that the results obtained will depend on the context in which an observable is studied.

(Standard) Quantum Mechanics in NON-CONTEXTUAL.

Gleason's Theorem.

Statistical Mixture

Gleason's Theorem.

Statistical Mixture

- Mixed State

Gleason’s Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of ”ensembles” of quantum systems:

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_D; p_1, p_2, \ldots, p_D\right).$$

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_D; p_1, p_2, \ldots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$.

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_D; p_1, p_2, \ldots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$. Obviously $0 \leq p_d \leq 1$ and $\sum_1^D p_d = 1$.

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \dots, \vec{\psi}_D; p_1, p_2, \dots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$. Obviously $0 \leq p_d \leq 1$ and $\sum_1^D p_d = 1$.

- Mixed State Operator:

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \dots, \vec{\psi}_D; p_1, p_2, \dots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$. Obviously $0 \le p_d \le 1$ and $\sum_1^D p_d = 1$.

- Mixed State Operator:

$$\hat{\rho} := \sum_{d=1}^{D} p_d \left|\psi_d\right\rangle \left\langle\psi_d\right| \equiv \sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \qquad [7]$$

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_D; p_1, p_2, \ldots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$. Obviously $0 \leq p_d \leq 1$ and $\sum_1^D p_d = 1$.

- Mixed State Operator:

$$\hat{\rho} := \sum_{d=1}^{D} p_d \left|\psi_d\right\rangle \left\langle\psi_d\right| \equiv \sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \qquad [7]$$

Called Density Matrix

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_D; p_1, p_2, \ldots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$. Obviously $0 \leq p_d \leq 1$ and $\sum_1^D p_d = 1$.

- Mixed State Operator:

$$\hat{\rho} := \sum_{d=1}^{D} p_d \left|\psi_d\right\rangle \left\langle\psi_d\right| \equiv \sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \qquad [7]$$

Called Density Matrix (Density Operator in infinite dimensions.)

Gleason's Theorem.

Statistical Mixture

- Mixed State

Need to generalize formalism of QM for the case of "ensembles" of quantum systems: Mixed State

$$\rho = \left(\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_D; p_1, p_2, \ldots, p_D\right).$$

Here, p_d is the "classical" probability that the system is in state $\vec{\psi}_d$. Obviously $0 \leq p_d \leq 1$ and $\sum_1^D p_d = 1$.

- Mixed State Operator:

$$\hat{\rho} := \sum_{d=1}^{D} p_d \left|\psi_d\right\rangle \left\langle\psi_d\right| \equiv \sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \qquad [7]$$

Called Density Matrix (Density Operator in infinite dimensions.)

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Probabilities. Suppose a state $\hat{\rho}$. Then

$$\begin{aligned}\mathrm{Prob}(A = a_n; \rho) &= \sum_{d=1}^{D} p_d \mathrm{Prob}\left(A = a_n; \left|\psi_d\right\rangle\right) \\ &= \sum_{d=1}^{D} p_d \left\langle \vec{\psi}_d \middle| \hat{P}_n \middle| \vec{\psi}_d \right\rangle\end{aligned}$$

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Probabilities. Suppose a state $\hat{\rho}$. Then

$$\begin{aligned}\text{Prob}(A = a_n; \rho) &= \sum_{d=1}^{D} p_d \text{Prob}\left(A = a_n; |\psi_d\rangle\right) \\ &= \sum_{d=1}^{D} p_d \left\langle \vec{\psi}_d \middle| \hat{P}_n \middle| \vec{\psi}_d \right\rangle\end{aligned}$$

But,

$$\begin{aligned}\left\langle \vec{\psi} \middle| \hat{B} \middle| \vec{\psi} \right\rangle &= \sum_{i=1}^{N} \left\langle \vec{\psi} \middle| \hat{B} \middle| e_i \right\rangle \left\langle e_i \middle| \vec{\psi} \right\rangle \\ &= \sum_{i=1}^{N} \left\langle e_i \middle| \vec{\psi} \right\rangle \left\langle \vec{\psi} \middle| \hat{B} \middle| e_i \right\rangle = \text{tr}\left(\hat{P}_\psi \hat{B}\right)\end{aligned}$$

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Probabilities. Suppose a state $\hat{\rho}$. Then

$$\begin{aligned}\mathrm{Prob}(A = a_n; \rho) &= \sum_{d=1}^{D} p_d \mathrm{Prob}\left(A = a_n; \left|\psi_d\right\rangle\right) \\ &= \sum_{d=1}^{D} p_d \left\langle\vec{\psi}_d\middle|\hat{P}_n\middle|\vec{\psi}_d\right\rangle\end{aligned}$$

But,

$$\begin{aligned}\left\langle\vec{\psi}\middle|\hat{B}\middle|\vec{\psi}\right\rangle &= \sum_{i=1}^{N} \left\langle\vec{\psi}\middle|\hat{B}\middle|e_i\right\rangle\left\langle e_i\middle|\vec{\psi}\right\rangle \\ &= \sum_{i=1}^{N} \left\langle e_i\middle|\vec{\psi}\right\rangle\left\langle\vec{\psi}\middle|\hat{B}\middle|e_i\right\rangle = \mathrm{tr}\left(\hat{P}_\psi \hat{B}\right)\end{aligned}$$

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Thus

$$\text{Prob}(A = a_n; \rho) = \sum_{d=1}^{D} p_d \,\text{tr}\Big(\hat{P}_{|\psi_d\rangle} \hat{P}_n\Big) = \text{tr}\Bigg(\sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \hat{P}_n\Bigg)$$
$$= \text{tr}\Big(\hat{\rho}\hat{P}_n\Big)$$

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Thus

$$\mathrm{Prob}(A = a_n; \rho) = \sum_{d=1}^{D} p_d \, \mathrm{tr}\Big(\hat{P}_{|\psi_d\rangle} \hat{P}_n\Big) = \mathrm{tr}\Big(\sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \hat{P}_n\Big)$$
$$= \mathrm{tr}\Big(\hat{\rho}\hat{P}_n\Big)$$

In a similar manner,

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Thus

$$\mathrm{Prob}(A = a_n; \rho) = \sum_{d=1}^{D} p_d \,\mathrm{tr}\Big(\hat{P}_{|\psi_d\rangle}\hat{P}_n\Big) = \mathrm{tr}\Bigg(\sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle}\hat{P}_n\Bigg)$$
$$= \mathrm{tr}\Big(\hat{\rho}\hat{P}_n\Big)$$

In a similar manner, expected value of A is

$$\langle A\rangle_\rho = \mathrm{tr}\Big(\hat{\rho}\hat{A}\Big)$$

Gleason's Theorem.

Density Operator: Probabilities and Expectation Values.

Thus

$$\mathrm{Prob}(A = a_n; \rho) = \sum_{d=1}^{D} p_d \, \mathrm{tr}\left(\hat{P}_{|\psi_d\rangle} \hat{P}_n\right) = \mathrm{tr}\left(\sum_{d=1}^{D} p_d \hat{P}_{|\psi_d\rangle} \hat{P}_n\right)$$
$$= \mathrm{tr}\left(\hat{\rho}\hat{P}_n\right)$$

In a similar manner, expected value of A is

$$\langle A \rangle_\rho = \mathrm{tr}\left(\hat{\rho}\hat{A}\right)$$

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^\dagger$;

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi| \hat{\rho} |\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)

Gleason’s Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^\dagger$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

Gleason’s Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^\dagger$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^\dagger$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace $\Rightarrow$ discete specturm

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace ⇒ discete specturm
2. Positive, semi-definite ⇒ non negative eigenvalues

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace ⇒ discete specturm
2. Positive, semi-definite ⇒ non negative eigenvalues
3. (1)+(2) ⇒ Always

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^\dagger$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace ⇒ discete specturm
2. Positive, semi-definite ⇒ non negative eigenvalues
3. (1)+(2) ⇒ Always

$$\hat{\rho} = \sum_n w_n \hat{P}_n,\ 0 \leq w_n \leq 1$$

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace ⇒ discete specturm
2. Positive, semi-definite ⇒ non negative eigenvalues
3. (1)+(2) ⇒ Always

$$\hat{\rho} = \sum_n w_n \hat{P}_n,\ 0 \leq w_n \leq 1$$

With eigenvalues w_n and relevant spectrtal projectors $\hat{P}_n$.

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace ⇒ discete specturm
2. Positive, semi-definite ⇒ non negative eigenvalues
3. (1)+(2) ⇒ Always

$$\hat{\rho} = \sum_n w_n \hat{P}_n,\ 0 \leq w_n \leq 1$$

With eigenvalues w_n and relevant spectrtal projectors $\hat{P}_n$.
Of course $\sum_n w_n = \mathrm{tr}(\hat{\rho}) = 1$.

Gleason's Theorem.

Density Operator Properties

In general, any operator with the properties

1. $\hat{\rho} = \hat{\rho}^{\dagger}$;
2. $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for all $|\psi\rangle$; (positive, semi-definite operator)
3. $\mathrm{Tr}(\hat{\rho}) = 1$

is called a density matrix (operator).

1. Finite trace $\Rightarrow$ discete specturm
2. Positive, semi-definite $\Rightarrow$ non negative eigenvalues
3. (1)+(2) $\Rightarrow$ Always

$$\hat{\rho} = \sum_{n} w_n \hat{P}_n,\ 0 \leq w_n \leq 1$$

With eigenvalues w_n and relevant spectrtal projectors $\hat{P}_n$.
Of course $\sum_n w_n = \mathrm{tr}(\hat{\rho}) = 1$.

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero,

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0.

Gleason’s Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\widehat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|\,.$$

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle \langle\psi| \,.$$

Thus, $\vec{\psi} \to |\psi\rangle \langle\psi|$

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|.$$

Thus, $\vec{\psi} \to |\psi\rangle\langle\psi|$

Interpretation of $\hat{\rho}$

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|.$$

Thus, $\vec{\psi} \rightarrow |\psi\rangle\langle\psi|$

Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|\,.$$

Thus, $\vec{\psi} \rightarrow |\psi\rangle\langle\psi|$
Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!
- The spectral decomposition is with respect to pairwise orthogonal projectors but this is not a requirement for a classical mixture of states.

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|.$$

Thus, $\vec{\psi} \to |\psi\rangle\langle\psi|$
Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!
- The spectral decomposition is with respect to pairwise orthogonal projectors but this is not a requirement for a classical mixture of states. Observe that in [7] the projectors $\hat{P}_{|\psi_d\rangle}$

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|\,.$$

Thus, $\vec{\psi} \rightarrow |\psi\rangle\langle\psi|$
Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!
- The spectral decomposition is with respect to pairwise orthogonal projectors but this is not a requirement for a classical mixture of states. Observe that in [7] the projectors $\hat{P}_{|\psi_d\rangle}$ need not be pairwise orthogonal.

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle \langle\psi| .$$

Thus, $\vec{\psi} \to |\psi\rangle \langle\psi|$
Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!
- The spectral decomposition is with respect to pairwise orthogonal projectors but this is not a requirement for a classical mixture of states. Observe that in [7] the projectors $\hat{P}_{|\psi_d\rangle}$ need not be pairwise orthogonal.

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|\,.$$

Thus, $\vec{\psi} \rightarrow |\psi\rangle\langle\psi|$
Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!
- The spectral decomposition is with respect to pairwise orthogonal projectors but this is not a requirement for a classical mixture of states. Observe that in [7] the projectors $\hat{P}_{|\psi_d\rangle}$ need not be pairwise orthogonal.

Gleason's Theorem.

Density Operator Properties

If only one eigenvalue is different from zero, then it is equal to 1 and all the others are equal to 0. Then the state is called Pure and is of the form

$$\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|.$$

Thus, $\vec{\psi} \to |\psi\rangle\langle\psi|$
Interpretation of $\hat{\rho}$

- Mixed State Decomposition [7] is not unique!
- The spectral decomposition is with respect to pairwise orthogonal projectors but this is not a requirement for a classical mixture of states. Observe that in [7] the projectors $\hat{P}_{|\psi_d\rangle}$ need not be pairwise orthogonal.

Gleason's Theorem.

Non Unique Decomposition Example: Spin-$\frac{1}{2}$ System

Gleason's Theorem.

Non Unique Decomposition Example: Spin-$\frac{1}{2}$ System

$$\hat{P}_{s_z=\hbar/2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{P}_{s_z=-\hbar/2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}_{s_x=\hbar/2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \hat{P}_{s_x=-\hbar/2} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Gleason's Theorem.

Non Unique Decomposition Example: Spin-$\frac{1}{2}$ System

$$\hat{P}_{s_z=\hbar/2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{P}_{s_z=-\hbar/2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}_{s_x=\hbar/2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \hat{P}_{s_x=-\hbar/2} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Take the density operator

$$\hat{\rho} := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Gleason's Theorem.

Non Unique Decomposition Example: Spin-$\frac{1}{2}$ System

$$\hat{P}_{s_z=\hbar/2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{P}_{s_z=-\hbar/2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}_{s_x=\hbar/2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \hat{P}_{s_x=-\hbar/2} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Take the density operator

$$\hat{\rho} := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Gleason’s Theorem.

Example Continues...

Gleason's Theorem.

Example Continues...

Then it decomposes,

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

or as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_x=\hbar/2} + \frac{1}{2}\hat{P}_{s_x=-\hbar/2}$$

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

or as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_x=\hbar/2} + \frac{1}{2}\hat{P}_{s_x=-\hbar/2}$$

We cannot identify the original mixture, by simply measuring, the spin!

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

or as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_x=\hbar/2} + \frac{1}{2}\hat{P}_{s_x=-\hbar/2}$$

We cannot identify the original mixture, by simply measuring, the spin!

Resulting Problem:

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

or as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_x=\hbar/2} + \frac{1}{2}\hat{P}_{s_x=-\hbar/2}$$

We cannot identify the original mixture, by simply measuring, the spin!

Resulting Problem: Not a simple interpretation of a density matrix as a measure of our ignorance of which pure states should be used to describe the system.

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

or as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_x=\hbar/2} + \frac{1}{2}\hat{P}_{s_x=-\hbar/2}$$

We cannot identify the original mixture, by simply measuring, the spin!

Resulting Problem: Not a simple interpretation of a density matrix as a measure of our ignorance of which pure states should be used to describe the system.

Gleason's Theorem.

Example Continues...

Then it decomposes, either as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_z=\hbar/2} + \frac{1}{2}\hat{P}_{s_z=-\hbar/2}$$

or as

$$\hat{\rho} = \frac{1}{2}\hat{P}_{s_x=\hbar/2} + \frac{1}{2}\hat{P}_{s_x=-\hbar/2}$$

We cannot identify the original mixture, by simply measuring, the spin!

Resulting Problem: Not a simple interpretation of a density matrix as a measure of our ignorance of which pure states should be used to describe the system.

Gleason's Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

Gleason’s Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

is again a density operator.

Gleason's Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

is again a density operator.

The only density matrices that cannot be decomposed into an affine sum

Gleason's Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

is again a density operator.
The only density matrices that cannot be decomposed into an affine sum are ones of the form

$$\hat{P}_\psi = |\psi\rangle \langle\psi|$$

Gleason's Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

is again a density operator.
The only density matrices that cannot be decomposed into an affine sum are ones of the form

$$\hat{P}_\psi = |\psi\rangle \langle\psi|$$

called pure states earlier.

Gleason's Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

is again a density operator.
The only density matrices that cannot be decomposed into an affine sum are ones of the form

$$\hat{P}_\psi = |\psi\rangle \langle\psi|$$

called pure states earlier. The rest are called mixed.

Gleason's Theorem.

Pure vs Mixed States

The set of density operators viewed as a subspace of all the bounded operators in $\mathscr{H}$ is convex, i.e., the affine sum

$$\sum_{i=1}^{K} r_i \hat{\rho}_i, \quad r_i > 0, \quad \sum_{i=1}^{K} r_i = 1$$

is again a density operator.
The only density matrices that cannot be decomposed into an affine sum are ones of the form

$$\hat{P}_\psi = |\psi\rangle \langle\psi|$$

called pure states earlier. The rest are called mixed.

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

- important:

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar\frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

- important:
  Evolution under the above law

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

- important:
  Evolution under the above law can never take a pure initial state into a mixed later state.

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

- important:
  Evolution under the above law can never take a pure initial state into a mixed later state. We specify the initial state $|\psi(0)\rangle$,

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

- important:
  Evolution under the above law can never take a pure initial state into a mixed later state. We specify the initial state $|\psi(0)\rangle$, and theory uniquely predicts state $|\psi(t)\rangle$, at a later time.

Gleason's Theorem.

Density Operator Time Evolution

- Density operators evolve in time according to

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[\hat{H},\ \hat{\rho}(t)\right]$$

- important:
  Evolution under the above law can never take a pure initial state into a mixed later state. We specify the initial state $|\psi(0)\rangle$, and theory uniquely predicts state $|\psi(t)\rangle$, at a later time.

Gleason's Theorem.

Question:

Gleason’s Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following ”elementary” fact:

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\mathrm{Prob}\,(P;\rho) = \mathrm{tr}\Big(\hat{\rho}\hat{P}\Big). \qquad [8]$$

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\text{Prob}\,(P;\rho) = \text{tr}\left(\hat{\rho}\hat{P}\right). \quad [8]$$

Question:

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\text{Prob}(P;\rho) = \text{tr}\Big(\hat{\rho}\hat{P}\Big). \qquad [8]$$

Question: Are there any other ways of assigning probabilities to the elements of $\mathscr{P}(\mathscr{H})$.

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\text{Prob}\,(P;\rho) = \text{tr}\Big(\hat{\rho}\hat{P}\Big). \qquad [8]$$

Question: Are there any other ways of assigning probabilities to the elements of $\mathscr{P}(\mathscr{H})$. i.e.,

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\mathrm{Prob}\,(P;\rho) = \mathrm{tr}\Big(\hat{\rho}\hat{P}\Big). \qquad [8]$$

Question: Are there any other ways of assigning probabilities to the elements of $\mathscr{P}(\mathscr{H})$. i.e., Is the standard quantum formalism unique or not?

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\mathrm{Prob}\,(P;\rho) = \mathrm{tr}\Big(\hat{\rho}\hat{P}\Big). \qquad [8]$$

Question: Are there any other ways of assigning probabilities to the elements of $\mathscr{P}(\mathscr{H})$. i.e., Is the standard quantum formalism unique or not?

Gleason's Theorem.

Question:

The probabilistic rules for QM, may be viewed as based on the following "elementary" fact: For any projector $\hat{P} \in \mathscr{P}(\mathscr{H})$ which corresponds a Proposition P:

$$\mathrm{Prob}\,(P;\rho) = \mathrm{tr}\Big(\hat{\rho}\hat{P}\Big). \quad [8]$$

Question: Are there any other ways of assigning probabilities to the elements of $\mathscr{P}(\mathscr{H})$. i.e., Is the standard quantum formalism unique or not?

Gleason’s Theorem.

Fundamental Assumptions

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

$$0 \leq \mathrm{Prob}\,(P) \leq 1 \;\; \text{for all} \;\; \hat{P} \in \mathscr{P}\,(\mathscr{H}) \tag{9}$$

$$\mathrm{Prob}\left(\hat{0}\right) = 0 \;\; \text{and} \;\; \mathrm{Prob}\left(\hat{\mathbb{1}}\right) = 1 \tag{10}$$

$$\mathrm{Prob}\left(\sum_{i=1}^{\infty} \hat{P}_i\right) = \sum_{i=1}^{\infty} \mathrm{Prob}\left(\hat{P}_i\right) \tag{11}$$

Where $\left\{\hat{P}_1, \hat{P}_1, \ldots\right\}$ can be any finite or countably infitine set of projection operators that are pairwise mutually orthogonal.

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

$$0 \leq \mathrm{Prob}\,(P) \leq 1 \;\; \text{for all} \;\; \hat{P} \in \mathscr{P}(\mathscr{H}) \tag{9}$$

$$\mathrm{Prob}\left(\sum_{i=1}^{\infty} \hat{P}_i\right) = \sum_{i=1}^{\infty} \mathrm{Prob}\left(\hat{P}_i\right) \tag{11}$$

Where $\left\{\hat{P}_1, \hat{P}_1, \ldots\right\}$ can be any finite or countably infitine set of projection operators that are pairwise mutually orthogonal.

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

$$0 \leq \mathrm{Prob}(P) \leq 1 \text{ for all } \hat{P} \in \mathscr{P}(\mathscr{H}) \quad [9]$$

$$\mathrm{Prob}\left(\hat{0}\right) = 0 \text{ and } \mathrm{Prob}\left(\hat{\mathbb{1}}\right) = 1 \quad [10]$$

$$\mathrm{Prob}\left(\sum_{i=1}^{\infty} \hat{P}_i\right) = \sum_{i=1}^{\infty} \mathrm{Prob}\left(\hat{P}_i\right) \quad [11]$$

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

$$0 \leq \mathrm{Prob}(P) \leq 1 \text{ for all } \hat{P} \in \mathscr{P}(\mathscr{H}) \quad [9]$$

$$\mathrm{Prob}\left(\hat{0}\right) = 0 \text{ and } \mathrm{Prob}\left(\hat{\mathbb{1}}\right) = 1 \quad [10]$$

$$\mathrm{Prob}\left(\sum_{i=1}^{\infty} \hat{P}_i\right) = \sum_{i=1}^{\infty} \mathrm{Prob}\left(\hat{P}_i\right) \quad [11]$$

Where $\left\{\hat{P}_1, \hat{P}_1, \ldots\right\}$ can be any finite or countably infitine set of projection operators that

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

$$0 \leq \mathrm{Prob}(P) \leq 1 \text{ for all } \hat{P} \in \mathscr{P}(\mathscr{H}) \quad [9]$$

$$\mathrm{Prob}\left(\hat{0}\right) = 0 \text{ and } \mathrm{Prob}\left(\hat{\mathbb{1}}\right) = 1 \quad [10]$$

$$\mathrm{Prob}\left(\sum_{i=1}^{\infty} \hat{P}_i\right) = \sum_{i=1}^{\infty} \mathrm{Prob}\left(\hat{P}_i\right) \quad [11]$$

Where $\left\{\hat{P}_1, \hat{P}_1, \ldots\right\}$ can be any finite or countably infitine set of projection operators that are pairwise mutually orthogonal.

Gleason's Theorem.

Fundamental Assumptions

A (minimal) set of requirements for defining a probability map $\mathscr{P}(\mathscr{H}) \Rightarrow \mathbb{R}$ is:

$$0 \leq \mathrm{Prob}(P) \leq 1 \quad \text{for all } \hat{P} \in \mathscr{P}(\mathscr{H}) \tag{9}$$

$$\mathrm{Prob}\left(\hat{0}\right) = 0 \quad \text{and} \quad \mathrm{Prob}\left(\hat{\mathbb{1}}\right) = 1 \tag{10}$$

$$\mathrm{Prob}\left(\sum_{i=1}^{\infty} \hat{P}_i\right) = \sum_{i=1}^{\infty} \mathrm{Prob}\left(\hat{P}_i\right) \tag{11}$$

Where $\left\{\hat{P}_1, \hat{P}_1, ...\right\}$ can be any finite or countably infitine set of projection operators that are pairwise mutually orthogonal.

Gleason's Theorem.

Statement - Implications

Gleason's Theorem.

Statement - Implications

Gleason (Gleason 1957) showed that, provided $\dim \mathscr{H} \geq 3$ the only way of satisfying Eq. [9] - [11] is with the aid of a density matrix, and Eq. [8].

Gleason's Theorem.

Statement - Implications

Gleason (Gleason 1957) showed that, provided $\dim \mathscr{H} \geq 3$ the only way of satisfying Eq. [9] - [11] is with the aid of a density matrix, and Eq. [8]. The proof is geometrical in nature and Gleason's theorem gave the motivation for the Kochen-Specker Theorem.

Gleason's Theorem.

Statement - Implications

Gleason (Gleason 1957) showed that, provided $\dim \mathscr{H} \geq 3$ the only way of satisfying Eq. [9] - [11] is with the aid of a density matrix, and Eq. [8]. The proof is geometrical in nature and Gleason's theorem gave the motivation for the Kochen-Specker Theorem.
If one wishes to construct hidden variable theories,

Gleason's Theorem.

Statement - Implications

Gleason (Gleason 1957) showed that, provided $\dim \mathscr{H} \geq 3$ the only way of satisfying Eq. [9] - [11] is with the aid of a density matrix, and Eq. [8]. The proof is geometrical in nature and Gleason's theorem gave the motivation for the Kochen-Specker Theorem. If one wishes to construct hidden variable theories, then Gleason's theorem places strong constraints on any attempts to modify the standard quantum formalism!

Gleason's Theorem.

Statement - Implications

Gleason (Gleason 1957) showed that, provided $\dim \mathscr{H} \geq 3$ the only way of satisfying Eq. [9] - [11] is with the aid of a density matrix, and Eq. [8]. The proof is geometrical in nature and Gleason's theorem gave the motivation for the Kochen-Specker Theorem.

If one wishes to construct hidden variable theories, then Gleason's theorem places strong constraints on any attempts to modify the standard quantum formalism!

The Kochen Specker theorem may be viewed as a corollary to Gleason's theorem!

Gleason's Theorem.

Statement - Implications

Gleason (Gleason 1957) showed that, provided $\dim \mathscr{H} \geq 3$ the only way of satisfying Eq. [9] - [11] is with the aid of a density matrix, and Eq. [8]. The proof is geometrical in nature and Gleason's theorem gave the motivation for the Kochen-Specker Theorem.

If one wishes to construct hidden variable theories, then Gleason's theorem places strong constraints on any attempts to modify the standard quantum formalism!

The Kochen Specker theorem may be viewed as a corollary to Gleason's theorem!

Other Important Theorems.

Other Important Theorems.

- (Bell 1964): On the Einstein Podolsky Rosen paradox. Also (Bell 1966), (N. D. Mermin 1981), (N. David Mermin 1990a), (N. David Mermin 1990b), (N. David Mermin 1993)
- (Sorkin 1993): Impossible Measurements on Quantum Fields
- (Haag 1955): ”Non Existence of Interaction Picture”. Also (Earman and Fraser 2006)

Other Important Theorems.

- (Bell 1964): On the Einstein Podolsky Rosen paradox. Also (Bell 1966), (N. D. Mermin 1981), (N. David Mermin 1990a), (N. David Mermin 1990b), (N. David Mermin 1993)
- (Sorkin 1993): Impossible Measurements on Quantum Fields
- (Haag 1955): ”Non Existence of Interaction Picture”. Also (Earman and Fraser 2006)

- Bell, J. S. (Nov. 1964). “On the Einstein Podolsky Rosen paradox.” In: Physics Physique Fizika 1 (3), pp. 195–200. DOI: 10.1103/PhysicsPhysiqueFizika.1.195.
- — (July 1966). “On the Problem of Hidden Variables in Quantum Mechanics.” In: Rev. Mod. Phys. 38 (3), pp. 447–452. DOI: 10.1103/RevModPhys.38.447.
- Earman, J. and D. Fraser (2006). “Haag’s Theorem and its Implications for the Foundations of Quantum Field Theory.” In: Erkenntnis 64.3, pp. 305–344. ISSN: 1572-8420. DOI: 10.1007/s10670-005-5814-y.
- Gleason, A. M. (1957). “Measures on the Closed Subspaces of a Hilbert Space.” In: Journal of Mathematics and Mechanics 6.6, pp. 885–893. ISSN: 00959057, 19435274.
- Haag, R. (1955). “On Quantum Field Theories.” In: Matematisk-fysiske Meddelelser 29.12.

- Isham, C. J. (1995). Lectures on Quantum Theory : Mathematical and Structural Foundations. Imperial College Press.
- Kochen, S. and E. P. Specker (1967). "The Problem of Hidden Variables in Quantum Mechanics." In: Journal of Mathematics and Mechanics 17.1, pp. 59–87. ISSN: 00959057, 19435274.
- Mermin, N. D. (Oct. 1981). "Bringing home the atomic world: Quantum mysteries for anybody." In: American Journal of Physics 49.10, pp. 940–943. ISSN: 0002-9505. DOI: 10.1119/1.12594. eprint: https://pubs.aip.org/aapt/ajp/article-pdf/49/10/940/11864850/940_1_online.pdf.
- Mermin, N. David (Aug. 1990a). "Quantum mysteries revisited." In: American Journal of Physics 58.8, pp. 731–734. ISSN: 0002-9505. DOI: 10.1119/1.16503.

eprint: https://pubs.aip.org/aapt/ajp/article-pdf/58/8/731/12179847/731_1_online.pdf.

Mermin, N. David (Dec. 1990b). "Simple unified form for the major no-hidden-variables theorems." In: Phys. Rev. Lett. 65 (27), pp. 3373–3376. DOI: 10.1103/PhysRevLett.65.3373.

— (July 1993). "Hidden variables and the two theorems of John Bell." In: Rev. Mod. Phys. 65 (3), pp. 803–815. DOI: 10.1103/RevModPhys.65.803.

Peres, A. (Feb. 1991). "Two simple proofs of the Kochen-Specker theorem." In: Journal of Physics A: Mathematical and General 24.4, p. L175. DOI: 10.1088/0305-4470/24/4/003.

— (2010). Quantum theory: concepts and methods. [Nachdr.] Fundamental theories of physics vol. 72. Dordrecht [u.a.]: Kluwer Acad. Publ. 446 pp. ISBN: 0792336321.

Hu, B. -L. and T. A. Jacobson, eds. (1993). Impossible Measurements on Quantum Fields. International Symposium on General Relativity. Cambridge: Cambridge University Press. 1369 pp. ISBN: 9780511524653.